**EXPOSITIONES MATHEMATICAE**

Expo. Math. 21 (2003): 63–74
© Urban & Fischer Verlag
www.urbanfischer.de/journals/expomath

Mathematical Note

# Hankel Determinants and Orthogonal Polynomials*

*Dedicated to Professor Alain M. Robert on the occasion of his retirement*

**Alexandre Junod**

Université de Neuchâtel, Institut de Mathématiques, Rue Émile-Argand 11, CH-2007 Neuchâtel, Suisse

**Abstract:** We develop a general context for the computation of the determinant of a Hankel matrix $H_n = (\alpha_{i+j})_{0\leqslant i,j\leqslant n}$, assuming some suitable conditions for the exponential (or ordinary) generating function of the sequence $(\alpha_n)_{n\geqslant 0}$. Several well-known particular cases are thus derived in a unified way.

## 1 Basic Observations

Let $\mathcal{A}$ be any integral domain i.e. a commutative ring with a unit element and no proper zero divisors. The underlying additive group acts by binomial convolution on the set

$$\mathcal{F}(\mathbb{N},\mathcal{A}) = \{\text{functions } \alpha : \mathbb{N} \longrightarrow \mathcal{A}\} = \{\text{sequences } (\alpha_n)_{n\geqslant 0} \subset \mathcal{A}\} :$$

For $a \in \mathcal{A}$ and $\alpha \in \mathcal{F}(\mathbb{N},\mathcal{A})$, we define $T^a\alpha \in \mathcal{F}(\mathbb{N},\mathcal{A})$ by

$$(T^a\alpha)_n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\alpha_k$$

and we check that $T^aT^b = T^{a+b}$. This action preserves the Hankel determinants :

**Proposition 1.** *If $H_n$ is the Hankel matrix (of order n) associated to $\alpha \in \mathcal{F}(\mathbb{N},\mathcal{A})$, then the Hankel matrix (of order n) associated to the convolution $T^a\alpha \in \mathcal{F}(\mathbb{N},\mathcal{A})$ is given by $H_{a,n} = SH_nS^t$ where $S = S_n(a) = \left(\binom{i}{j}a^{i-j}\right)_{0\leqslant i,j\leqslant n}$ is triangular with 1's on the diagonal. In particular,* $\det H_{a,n} = \det H_n$ *is independent of $a \in \mathcal{A}$.*

PROOF. By direct calculation, we have

$$(SHS^t)_{ij} = \sum_{k=0}^{n} S_{jk} \sum_{l=0}^{n} S_{il}H_{lk} = \sum_{k\geqslant 0} \binom{j}{k} a^{j-k} \sum_{l\geqslant 0} \binom{i}{l} a^{i-l}\alpha_{l+k}.$$

*Supported by the Swiss National Science Foundation, grant 20-56816.99

0732-0869/03/21/1-63 $15.00/0

Setting $m = k + l$ and using the Vandermonde convolution formula, we get

$$(SHS^t)_{ij} = \sum_{m\geqslant 0}\sum_{k=0}^{m}\binom{j}{k}\binom{i}{m-k}a^{i+j-m}\alpha_m = \sum_{m\geqslant 0}\binom{i+j}{m}a^{i+j-m}\alpha_m,$$

in other words, $(SHS^t)_{ij} = (T^a\alpha)_{i+j} = (H_{a,n})_{ij}$. $\square$

Notice that, for any element $a \in \mathcal{A}$, the matrix $S_n(a)$ is lower triangular and we have $S_n(a)S_n(b) = S_n(a+b)$, in particular $S_n(a) = S_n(1)^a$ when $a \in \mathbb{Z}$.

The miscellaneous works of C. Radoux [4],[5] lead to the following result.

**Proposition 2.** *Denote by $F(z) = \sum \alpha_n \frac{z^n}{n!}$ the exponential generating function of a sequence $\alpha \in \mathcal{F}(\mathbb{N}, \mathcal{A})$ and by $\partial = \partial_z$ the derivative with respect to the variable $z$. If there exist formal series $F_k(z) \in \mathcal{A}[[z]]$ and elements $d_k \in \mathcal{A}$ such that*

1) $\left[\partial^n F_k(z)\right]_{z=0} = 0$ *whenever $k > n$,*
2) $\sum_{k\geqslant 0} d_k F_k(y) F_k(z) = F(y+z)$,

*then the Hankel determinants are given by* $\det H_n = \prod_{k=0}^{n} d_k \left[\partial^k F_k(z)\right]_{z=0}^2$.

PROOF. With the binomial identity, we can write

$$F(y+z) = \sum_{n\geqslant 0}\alpha_n \frac{(y+z)^n}{n!} = \sum_{m,n\geqslant 0}\alpha_{m+n}\frac{y^n}{n!}\frac{z^m}{m!}$$

while the Taylor expansions

$$F_k(y) = \sum_{n\geqslant 0}\partial^n F_k(0)\frac{y^n}{n!} \quad \text{and} \quad F_k(z) = \sum_{m\geqslant 0}\partial^m F_k(0)\frac{z^m}{m!}$$

allow us to write

$$\sum_{k\geqslant 0} d_k F_k(y) F_k(z) = \sum_{m,n\geqslant 0}\sum_{k\geqslant 0} d_k\ \partial^n F_k(0)\ \partial^m F_k(0)\ \frac{y^n}{n!}\frac{z^m}{m!}.$$

By identification, the second condition expresses the fact that

$$\alpha_{m+n} = \sum_{k\geqslant 0} d_k \left[\partial^n F_k(z)\right]_{z=0}\left[\partial^m F_k(z)\right]_{z=0}.$$

Thus the Hankel matrix $H_n = (\alpha_{i+j})_{0\leqslant i,j\leqslant n}$ admits a decomposition $H_n = L_n D_n L_n^t$ where $D_n = \mathrm{diag}(d_0, d_1, \ldots, d_n)$ is a diagonal matrix and $L_n = \left(\partial^i F_j(z)\big|_{z=0}\right)_{0\leqslant i,j\leqslant n}$ a lower triangular matrix, due to the first condition. The proposition is now obvious. $\square$

## An important case

If $\mathcal{A}$ is a polynomial ring, we can consider sequences $(P_n(x))_{n\geqslant 0}$ where $\deg P_n = n$. The exponential generating function of such a sequence is then denoted by $F(x,z) = \sum_{n\geqslant 0} P_n(x)\frac{z^n}{n!}$ and in many cases, we can take $F_k(x,z) = \partial_x^k F(x,z)$ where $\partial_x = \partial/\partial x$. The first condition of the proposition is automatically satisfied :

$$\Big[\partial_z^n F_k(x,z)\Big]_{z=0} = \Big[\partial_z^n \partial_x^k F(x,z)\Big]_{z=0} = \partial_x^k \Big[\partial_z^n F(x,z)\Big]_{z=0} = \partial_x^k P_n(x)$$

vanishes whenever $k > n$. So, if there exist polynomials $d_k(x)$ such that

$$\sum_{k\geqslant 0} d_k(x)\partial_x^k F(x,y)\partial_x^k F(x,z) = F(x,y+z), \qquad (*)$$

then the Hankel determinants are $\det(P_{i+j}(x))_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} d_k(x)\Big(\partial_x^k P_k(x)\Big)^2$.

## Examples

**1)** The simplest example is given by the *canonical basis* $P_n(x) = x^n$ : we have $F(x,z) = e^{xz}$ and the relation $(*)$ can be written $\sum_{k\geqslant 0} d_k(x)(yz)^k = 1$. The constant polynomials $d_0(x) = 1$, $d_k(x) = 0$ if $k \geqslant 1$ are suitable and the proposition confirms the obvious result

$$\det(x^{i+j})_{0\leqslant i,j\leqslant n} = d_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

**2)** For the polynomials $P_n(x) = n!x^n$, we have the generating function $F(x,z) = (1-zx)^{-1}$ and we find $\partial_x^k F(x,z) = k!z^k(1-zx)^{-k-1}$ (for $k \geqslant 0$). The relation $(*)$ becomes

$$\sum_{k\geqslant 0} d_k(x)(k!)^2(yz)^k[1 - x(y+z) + x^2yz]^{-k-1} = (1 - x(y+z))^{-1}.$$

Now we take $d_k(x) = (x^k/k!)^2$ and obtain

$$\det((i+j)!x^{i+j})_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} (x^k/k!)^2(k!)^4 = \prod_{k=0}^{n} (k!x^k)^2 = \Big(\prod_{k=0}^{n} k!\Big)^2 x^{n(n+1)}.$$

**3)** The polynomials $D_n(x) = \sum_{k=0}^{n} \binom{n}{k} k!(-1)^{n-k}x^k$ generate the same determinants : the exponential generating function is $F(x,z) = e^{-z}(1-xz)^{-1}$ and the relation $(*)$ can be written exactly as above. More generally, the first proposition shows that the Hankel determinants generated by a sequence

$$P_{a,n}(x) = \sum_{k=0}^{n} \binom{n}{k}(-a)^{n-k}P_k(x)$$

are independant of $a \in \mathbb{R}$ and coincide with those for the sequence $P_{0,n}(x) = P_n(x)$. The polynomials $D_n(x)$ are obtained from $P_n(x) = n!x^n$ and $a = 1$. Their study is interesting because $D_n(1)$ is the *derangement number* (i.e. number of permutations without fixed point) of a set of $n$ elements.

**4)** Let $I_n$ be the *number of involutions* of a set of $n$ elements, that is the number of permutations $\sigma \in Sym(n)$ of order two. We have $I_0 = I_1 = 1$ and the recurrence relation $I_{n+1} = I_n + nI_{n-1}$ leads to the generating function $\sum I_n \frac{z^n}{n!} = e^{z+z^2/2}$. To use the corollary, we have to construct some polynomials $I_n(x)$ taking the special value $I_n$. It seems natural to consider $\sum I_n(x)\frac{z^n}{n!} = e^{xz+z^2/2} =: F(x,z)$ leading to $I_n(1) = I_n$. Derivating $F(x,z)$ respectively to $z$, we find $I_{n+1}(x) = xI_n(x) + nI_{n-1}(x)$, which shows that the $I_n(x)$ are monic polynomials with $\deg I_n(x) = n$. The relation $(*)$ becomes

$$\sum_{k\geqslant 0} d_k(x)(yz)^k = e^{yz},$$

and with $d_k(x) = 1/k!$, the proposition gives

$$\det(I_{i+j}(x))_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} (1/k!)(k!)^2 = \prod_{k=0}^{n} k! = \det(I_{i+j})_{0\leqslant i,j\leqslant n}.$$

**5)** The *Hermite polynomials* $H_n(x)$ lead to the generating function $F(x,z) = e^{2xz-z^2}$. The relation $(*)$ is

$$\sum_{k\geqslant 0} d_k(x)(4yz)^k = e^{-2yz}.$$

It is satisfied by the constants $d_k(x) = 1/((-2)^k k!)$. Since the leading coefficient of $H_k(x)$ is $2^k$ (coming from the recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$), we have

$$\det(H_{i+j}(x))_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} \frac{1}{(-2)^k k!}(2^k k!)^2 = \prod_{k=0}^{n} k!(-2)^k = \Big(\prod_{k=0}^{n} k!\Big)(-2)^{n(n+1)/2}.$$

**6)** Consider now the *Bell polynomials* $B_n(x)$. We have $F(x,z) = e^{xg(z)}$ with $g(z) = e^z - 1$, and $(*)$ becomes

$$\sum_{k\geqslant 0} d_k(x)(e^{y+z} - e^y - e^z + 1)^k = e^{x(e^{y+z}-e^y-e^z+1)}.$$

This is verified for $d_k(x) = x^k/k!$ and we get

$$\det(B_{i+j}(x))_{0\leqslant i,j\leqslant n} = \prod_{k=0}^{n} (x^k/k!)(k!)^2 = \prod_{k=0}^{n} k!x^k = \Big(\prod_{k=0}^{n} k!\Big)x^{n(n+1)/2}.$$

7) The *Euler numbers* $E_n$ can be defined by $F(x) = \sum E_n \frac{z^n}{n!} = 1/\cos z$. This exponential generating function satisfies

$$F(y+z) = \frac{1}{\cos(y+z)} = \frac{\cos y \cos z}{\cos y \cos z - \sin y \sin z} \, \frac{1}{\cos y \cos z} = \frac{(1-\tan y \tan z)^{-1}}{\cos y \cos z}$$

and, using a geometric series, we can formally write

$$F(y+z) = \frac{1}{\cos y \cos z} \sum_{k\geqslant 0} (\tan y)^k (\tan z)^k = \sum_{k\geqslant 0} \frac{(\tan y)^k}{\cos y} \frac{(\tan z)^k}{\cos z}.$$

The second condition of the proposition is also fulfilled if we consider the constants $d_k = 1$ and the functions $F_k(z) = (\tan z)^k / \cos z$. The first condition is also satisfied :

$$E_{n,k} := \left[\partial^n F_k(z)\right]_{z=0} = \left[\partial^n \frac{(\tan z)^k}{\cos z}\right]_{z=0} \quad \text{is zero whenever } k > n \text{ and is } n! \text{ if } k = n.$$

These facts are obvious for $n = 0$ and follow by induction from the recurrence relation $E_{n+1,k} = kE_{n,k-1} + (k-1)E_{n,k+1}$. The proposition 2 finally gives

$$\det(E_{i+j})_{0\leqslant i,j\leqslant n} = \Big(\prod_{k=0}^{n} k!\Big)^2.$$

With the definition above, the Euler numbers of odd index are zero. For $\widetilde{E}_n = E_{2n}$, we find

$$\det(\widetilde{E}_{i+j})_{0\leqslant i,j\leqslant n} = \det(E_{2(i+j)})_{0\leqslant i,j\leqslant n} = \Big(\prod_{k=0}^{n}(2k)!\Big)^2$$

by adapting suitably the end of the proof of the proposition.

## 2 Exponential generating functions

In the example of Euler numbers (whose generating function is $F(z) = 1/\cos z$), we have introduced the functions $F_k(z) = g(z)^k F(z)$ where $g(z) = \tan z$ satisfies the differential equation $g'(z) = 1 + g(z)^2$ with initial condition $g(0) = 0$. This situation can be generalized as follows.

**Theorem 1.** *Consider any exponential generating function $F(z) = \exp G(z)$ where $G(0) = 0$ and $g(z) = G'(z) - G'(0)$ satisfies $g'(z) = \alpha + \beta g(z) + \gamma g(z)^2$ for some parameters $\alpha \neq 0$, $\beta$ and $\gamma$ in $\mathcal{A}$. Then*

$$F(y+z) = \sum_{k\geqslant 0} \frac{1\cdot(1+\gamma)\cdots(1+(k-1)\gamma)}{k!\alpha^k} \, g(y)^k F(y) \; g(z)^k F(z).$$

*The corresponding Hankel determinants are given by*

$$\det H_n = \alpha^{n(n+1)/2} \prod_{k=0}^{n} \Big(k!(1+\gamma)\cdots(1+(k-1)\gamma)\Big).$$

PROOF. Let us consider the functions $F_k(z) = g(z)^k F(z)$ (for $k \geqslant 0$). We can write

$$\begin{aligned} \partial F_k &= kg^{k-1}g'F + g^kF' \\ &= kg^{k-1}(\alpha+\beta g+\gamma g^2)F + g^kG'F \\ &= \big(k\alpha g^{k-1} + (G'(0)+k\beta)g^k + (1+k\gamma)g^{k+1}\big)F \\ &= k\alpha F_{k-1} + (G'(0)+k\beta)F_k + (1+k\gamma)F_{k+1} \end{aligned}$$

(by convention, $F_k(z) = 0$ for $k < 0$), that is to say

$$\partial^{n+1}F_k(z) = \partial^n\big[k\alpha F_{k-1}(z) + (G'(0)+k\beta)F_k(z) + (1+k\gamma)F_{k+1}(z)\big] \quad \text{for } n \geqslant 0.$$

This relation allows us to establish inductively the following facts (obvious for $n = 0$) :

- $\big[\partial^n F_k(z)\big]_{z=0} = 0$ whenever $k > n$,
- $\big[\partial^n F_n(z)\big]_{z=0} = n\alpha\big[\partial^{n-1}F_{n-1}(z)\big]_{z=0} = \cdots = n!\alpha^n$.

A direct calculation also shows that

$$\partial^{n+1}F_k(z)\ \partial^m F_k(z) - \partial^n F_k(z)\ \partial^{m+1}F_k(z) = k\alpha H_{k-1}(z) - (1+k\gamma)H_k(z)$$

with $H_k(z) = \partial^m F_{k+1}(z)\ \partial^n F_k(z) - \partial^m F_k(z)\ \partial^n F_{k+1}(z)$ $(= 0$ if $k < 0)$.
Consider now the elements (in Frac$\mathcal{A}$, field of fractions of $\mathcal{A}$)

$$d_k = \frac{(1+\gamma)\cdots(1+(k-1)\gamma)}{k!\alpha^k} \quad (= 1 \text{ if } k = 0).$$

They satisfy $d_{k+1}(k+1)\alpha = d_k(1+k\gamma)$, so that all terms in the sum

$$\sum_{k\geqslant 0} d_k\big[\partial^{n+1}F_k(z)\ \partial^m F_k(z) - \partial^n F_k(z)\ \partial^{m+1}F_k(z)\big] = \sum_{k\geqslant 0} d_k[k\alpha H_{k-1}(z) - (1+k\gamma)H_k(z)]$$

compensate each other. Thus, for all integers $m, n \geqslant 0$, we have

$$\sum d_k\partial^n F_k(z)\ \partial^m F_k(z) = \sum d_k\partial^{n-1}F_k(z)\ \partial^{m+1}F_k(z) = \cdots = \sum d_k F_k(z)\ \partial^{m+n}F_k(z)$$

and, evaluating this expression at $z = 0$, we get

$$\sum d_k\big[\partial^n F_k(z)\ \partial^m F_k(z)\big]_{z=0} = \sum d_k F_k(0)\big[\partial^{m+n}F_k(z)\big]_{z=0} = \big[\partial^{m+n}F(z)\big]_{z=0}.$$

This shows that, for the sequence corresponding to the generating function $F(z)$, we can use the last proposition with the elements $d_k$ and the functions $F_k(z)$ defined above. □

Here are the choices corresponding to the previous examples (1-7)

| | $F(z)$ | $G(z)$ | $g(z)$ | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|---|---|---|
| Canonical basis | $e^{xz}$ | $xz$ | $x$ | 0 | 0 | 0 |
| $P_n(x) = n!x^n$ | $(1-xz)^{-1}$ | $-\log(1-xz)$ | $\frac{x}{1-xz} - x$ | $x^2$ | $2x$ | 1 |
| Derangement polynomials | $e^{-z}(1-xz)^{-1}$ | $-z-\log(1-xz)$ | $\frac{x}{1-xz} - x$ | $x^2$ | $2x$ | 1 |
| Involution polynomials | $e^{xz+z^2/2}$ | $xz + \frac{z^2}{2}$ | $z$ | 1 | 0 | 0 |
| Hermite polynomials | $e^{2xz-z^2}$ | $2xz - z^2$ | $-2z$ | $-2$ | 0 | 0 |
| Bell polynomials | $e^{x(e^z-1)}$ | $x(e^z-1)$ | $x(e^z-1)$ | $x$ | 1 | 0 |
| Euler numbers | $1/\cos z$ | $-\log(\cos z)$ | $\tan z$ | 1 | 0 | 1 |

They may be supplemented by the following ones.

8) The polynomials $\widehat{B}_n(x)$ with exponential generating function $F(x,z) = \exp\left(\frac{e^{xz}-1}{x}\right)$ give the same Hankel determinants as the Bell polynomials $B_n(x)$.

9) The *Euler polynomials of order* $m \geqslant 1$ (denoted by $E_n^m(x)$) are defined by the generating function $F(x,z) = \left(\frac{2}{e^z+1}\right)^m e^{xz}$. By proposition 1, one can just consider the generating function $F(z) = F(0,z)$ to evaluate the Hankel determinants. We have $g(z) = m\left(\frac{1}{e^z+1} - \frac{1}{2}\right)$ and we can apply the theorem with $\alpha = -\frac{m}{4}$, $\beta = 0$ and $\gamma = \frac{1}{m}$ :

$$\det H_n(x) = \left(-\frac{1}{4}\right)^{n(n+1)/2} \prod_{k=0}^{n} k!m(m+1)\cdots(m+(k-1)) = \det H_n(0).$$

The reader may derive the value of the Hankel determinants associated to the ordinary Euler numbers $E_n$ (example 7) from the observation $E_n = (-2)^n E_n^1(\frac{1}{2})$. The results of [1], [2], [3], [4] and [5] are thus derived in a unified way.

# 3 Ordinary generating functions

The preceding theorem, concerning exponential generating functions (or "Hurwitz series"), can be translated for ordinary generating functions $F(z) = \sum \alpha_k z^k$. Having considered Hurwitz series $F(z) = \exp G(z) = \sum \frac{G(z)^k}{k!}$ with $G(0) = 0$, we can similarly consider ordinary generating function under the form

$$F(z) = \sum G(z)^k = \frac{1}{1-G(z)} \quad \text{with } G(0) = 0.$$

The derivation operator $\partial$ played an important part because it is *"associated"* to the polynomial basis $(f_n(z) = z^n/n!)$, in the sense that

$$\partial f_0(z) = 0 \quad \text{and} \quad \partial f_n(z) = f_{n-1}(z) \quad \text{for } n \geqslant 1.$$

Its analog in our new context is then given by $\nabla : f(z) \longmapsto \frac{f(z)-f(0)}{z}$, operator associated to the canonical basis $(z^n)$.

**Theorem 2.** *Consider an ordinary generating function $F(z) = \frac{1}{1-G(z)}$ with $G(0) = 0$ and suppose that $g(z) = \nabla G(z) - \nabla G(0) = \frac{G(z)}{z} - G'(0)$ satisfies $g(z) = z(\alpha + \beta g(z) + \gamma g(z)^2)$ for some parameters $\alpha \neq 0$, $\beta$ and $\gamma$ in $\mathcal{A}$. Then the Hankel determinants are given by*

$$\det H_n = \alpha^{n(n+1)/2} \gamma^{n(n-1)/2}.$$

PROOF. The functions $F_k(z) = g(z)^k F(z)$ satisfy $\nabla F_0(z) = G'(0)F_0(z) + F_1(z)$ and

$$\nabla F_k(z) = \alpha F_{k-1}(z) + \beta F_k(z) + \gamma F_{k+1}(z) \text{ for all } k \geqslant 1.$$

We establish by induction

$$\left[\nabla^n F_k(z)\right]_{z=0} = 0 \quad \text{for } k > n \qquad \text{and} \qquad \left[\nabla^n F_n(z)\right]_{z=0} = \alpha^n.$$

On the other hand, we show directly

$$\nabla^{n+1} F_k(z)\ \nabla^m F_k(z) - \nabla^n F_k(z)\ \nabla^{m+1} F_k(z) = \begin{cases} -H_0(z) & \text{if } k = 0 \\ \alpha H_{k-1}(z) - \gamma H_k(z) & \text{if } k \geqslant 1 \end{cases}$$

with $H_k(z) = \nabla^m F_{k+1}(z)\ \nabla^n F_k(z) - \nabla^m F_k(z)\ \nabla^n F_{k+1}(z)$. If we consider the elements $d_0 = 1$ and $d_k = \gamma^{k-1}/\alpha^k$ for $k \geqslant 1$, we see that

$$\sum_{k \geqslant 0} d_k \left[\nabla^{n+1} F_k(z)\ \nabla^m F_k(z) - \nabla^n F_k(z)\ \nabla^{m+1} F_k(z)\right]$$

is a vanishing telescoping sum. For all integers $m, n \geqslant 0$, we can write

$$\sum d_k \nabla^n F_k(z)\ \nabla^m F_k(z) = \sum d_k \nabla^{n-1} F_k(z)\ \nabla^{m+1} F_k(z) = \cdots = \sum d_k F_k(z)\ \nabla^{m+n} F_k(z)$$

and evaluation at $z = 0$ gives $\sum d_k \left[\nabla^n F_k(z)\ \nabla^m F_k(z)\right]_{z=0} = \left[\nabla^{m+n} F(z)\right]_{z=0}$. We conclude as in the proof of proposition 2. $\square$

**Remark**

Since $\frac{G(z)}{z} = \frac{1}{z}(1 - \frac{1}{F(z)}) = \frac{F(z)-1}{zF(z)} \underset{z\to 0}{\longrightarrow} F'(0)$, we have $g(z) = \frac{F(z)-1}{zF(z)} - F'(0)$ and it follows that $F(z) = (1 - F'(0)z - zg(z))^{-1}$. Moreover, the initial condition $g(0) = 0$ and the quadratic relation $g(z) = z(\alpha + \beta g(z) + \gamma g(z)^2)$ show that

$$g(z) = \frac{1 - \beta z - \sqrt{(1-\beta z)^2 - 4\gamma\alpha z^2}}{2\gamma z} \qquad (\gamma \neq 0).$$

**Examples**

**1.** The *Catalan numbers* correspond to the ordinary generating function $F(z) = \frac{1-\sqrt{1-4z}}{2z}$ and we have

$$g(z) = \frac{1 - \sqrt{1-4z} - 2z}{z(1-\sqrt{1-4z})} - 1 = \frac{4z - 2z(1+\sqrt{1-4z})}{4z^2} - 1 = \frac{1 - 2z - \sqrt{1-4z}}{2z}.$$

We can use the theorem with $\gamma = \alpha = 1$ and $\beta = 2$ : we obtain $\det H_n = 1$ for all $n \geqslant 0$. The *Motzkin numbers*, defined by the generating function $F(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$, admit exactly the same Hankel determinants (with the parameters $\gamma = \beta = \alpha = 1$).

**2.** The *Legendre polynomials* are defined by $F(x,z) = (1-2xz+z^2)^{-1/2}$ and the theorem can be used with $\alpha = \frac{x^2-1}{2}$, $\beta = x$ and $\gamma = \frac{1}{2}$. Here, we find

$$\det H_n(x) = \frac{(x^2-1)^{n(n+1)/2}}{2^{n^2}}.$$

# 4 Orthogonal polynomials

We suppose now that $\mathcal{A}$ is a subring of $\mathbb{R}$ and use the context of symbolic calculus : for a fixed sequence $\alpha \in \mathcal{F}(\mathbb{N}, \mathcal{A})$, we consider the $\mathcal{A}$-linear map

$$\Phi : \mathcal{A}[x] \longrightarrow \mathcal{A},\ x^n \longmapsto \alpha_n.$$

We suppose moreover that the Hankel determinants are all positive. That means that, for a fixed integer $n$, the symmetric matrix $H_n$ is positive-definite and that the bilinear map

$$(f,g) \longmapsto (f \mid g) := \Phi(f(x)g(x))$$

is an inner product on $V_n[x] := \{P(x) \in \mathcal{A}[x] : \deg P \leqslant n\}$ (the Gram matrix in the canonical basis is given by the Hankel matrix $H_n$). Since it is valid for all integer $n \geqslant 0$, this map is also an inner product on $\mathcal{A}[x] = \bigcup V_n[x]$.

An orthogonal system of monic polynomials is obtained with the Gram-Schmidt orthogonalization process :

$$P_0(x) = 1 \quad , \quad P_n(x) = \frac{1}{\det H_{n-1}} \begin{vmatrix} \alpha_0 & \cdots & \alpha_{n-1} & 1 \\ \alpha_1 & \cdots & \alpha_n & x \\ \vdots & & \vdots & \vdots \\ \alpha_n & \cdots & \alpha_{2n-1} & x^n \end{vmatrix} \quad \text{for } n \geqslant 1$$

and we note $P_n(x) = p_{n,0} + p_{n,1}x + p_{n,2}x^2 + \cdots + p_{n,n-1}x^{n-1} + x^n$ $(p_{n,n} = 1)$. These monic polynomials have coefficients in Frac$\mathcal{A}$ and verify a three-terms recurrence relation [6]

$$P_{n+1}(x) = (x - \lambda_n)P_n(x) - \mu_n P_{n-1}(x) \qquad (\text{for } n \geqslant 1)$$

with $\lambda_n = p_{n,n-1} - p_{n+1,n}$ and $\mu_n = \|P_n(x)\|^2/\|P_{n-1}(x)\|^2$.

Since the Hankel matrix (of a certain order $n$) is symmetric positive-definite, there is a unique decomposition $H = LDL^t$ where $L$ is a lower triangular matrix with 1's on the main diagonal and $D = \operatorname{diag}(d_0, d_1, \ldots, d_n)$ is a diagonal matrix. The relation $L^{-1}H(L^{-1})^t = D$ shows that

$$L^{-1} = \begin{pmatrix} p_{0,0} & 0 & \cdots & 0 \\ p_{1,0} & p_{1,1} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ p_{n,0} & p_{n,1} & \cdots & p_{n,n} \end{pmatrix}.$$

The $k$-th row of $L^{-1}$ consists in the coefficients of $P_k(x)$ and we have $\|P_k(x)\|^2 = d_k$. Hence $\mu_k = d_k/d_{k-1}$ and $\det H_n = \|P_0(x)\|^2 \cdot \|P_1(x)\|^2 \cdots \|P_n(x)\|^2$.

The matrix $M = L^{-1}$ has characteristic polynomial $P_M(x) = (1-x)^{n+1}$ and the Hamilton-Cayley theorem allows us to express formally (with the operator $\nabla$ defined above)

$$L = M^{-1} = -\frac{1}{\det M}\left[\nabla P_M(x)\right]_{x=M} = \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} M^{k-1}.$$

Considering the coefficients just under the main diagonal in $M, M^2, \ldots, M^n$, we see that $L_{k+1,k} = -p_{k+1,k}$, so that $\lambda_k = L_{k+1,k} - L_{k,k-1}$. We finally get the following result.

**Theorem 3.** *If the Hankel determinants are all positive and are evaluated with the exponential (resp. ordinary) generating function (theorem 1 resp. 2) then*

$$\lambda_n = G'(0) + n\beta \;\; \textit{and} \;\; \mu_n = n\alpha(1 + (n-1)\gamma) \quad \textit{for all } n \geqslant 1$$

$$(\textit{resp.} \;\; \lambda_1 = \beta, \; \mu_1 = \alpha \;\; \textit{and} \;\; \lambda_n = \beta, \; \mu_n = \alpha\gamma \quad \textit{for all } n \geqslant 2)$$

PROOF. With the previous notations and the desired normalization, we have

$$\mu_n = \frac{d_n\left[\partial^n F_n(z)\right]_{z=0}^2}{d_{n-1}\left[\partial^{n-1} F_{n-1}(z)\right]_{z=0}^2} = n\alpha(1 + (n-1)\gamma)$$

$$\lambda_n = \left[\frac{\partial^{n+1} F_n(z)}{\partial^n F_n(z)} - \frac{\partial^n F_{n-1}(z)}{\partial^{n-1} F_{n-1}(z)}\right]_{z=0} = \frac{1}{n!\alpha^n}\left[\partial^{n+1} F_n(z) - n\alpha\partial^n F_{n-1}(z)\right]_{z=0}.$$

By the recurrence relation obtained in the proof of theorem 1, we get $\lambda_n = G'(0) + n\beta$. The other case is similar. □

## Examples

**1.** We consider the linear map $\Phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]$ which sends the Pochhammer polynomial $(x)_n = x(x-1)\cdots(x+1-n)$ on $x^n$ and define the Bell polynomials by $B_n(x) = \Phi(x^n)$. Let $a > 0$ be a positive integer. We have seen that all Hankel determinants associated to the sequence $(B_n(a))$ are positive, so that this sequence generates an inner product $(f \mid g)_a = \Phi_a(fg) := \Phi(fg)\big|_{x=a}$ on $\mathbb{Z}[x]$. The monic polynomials recursively defined by $P_{a,0}(x) = 1,\ P_{a,1}(x) = x - a$ and the relation

$$P_{a,n+1}(x) = (x - a - n)P_{a,n}(x) - naP_{a,n-1}(x) \quad (n \geqslant 1)$$

form an orthogonal system in $\mathbb{Z}[x]$ (use the theorem 3 with $\alpha = a, \beta = 1$ and $\gamma = 0$). These polynomials, known as the *Charlier polynomials*, are

$$P_{a,n}(x) = \Phi^{-1}((x-a)^n) = \sum_{k=0}^{n} \binom{n}{k}(-a)^{n-k}(x)_k$$

and with the previously defined convolution operator, we have the commutative diagram

$$\begin{tikzcd}
(x)_n \arrow[r, "\Phi"] \arrow[d, shift left, "T^{-a}"] \arrow[dr, phantom, "\circlearrowleft"] & x^n \arrow[r, "\Phi"] \arrow[d, shift left, "T^{-a}"] \arrow[dr, phantom, "\circlearrowleft"] & B_n(x) \arrow[d, shift left, "T^{-a}"] \\
P_{a,n}(x) \arrow[r, "\Phi"] \arrow[u, shift left, "T^{a}"] & (x-a)^n \arrow[r, "\Phi"] \arrow[u, shift left, "T^{a}"] & B_{a,n}(x) \arrow[u, shift left, "T^{a}"]
\end{tikzcd}$$

**2.** The Euler numbers, defined by the exponential generating function $F(z) = 1/\cos z$, also furnish an inner product on $\mathbb{Z}[x]$ and the monic polynomials

$$Q_0(x) = 1,\ Q_1(x) = x \ \text{ and } Q_{n+1}(x) = xQ_n(x) - n^2 Q_{n-1}(x) \quad (n \geqslant 1)$$

(called *Meixner polynomials*) form an orthogonal system in $\mathbb{Z}[x]$ (use the last theorem with $\alpha = \gamma = 1$ and $\beta = 0$).

**3.** The monic polynomials which form an orthogonal system for the inner product associated to the sequence $n! = [n!x^n]_{x=1}$ are given by $L_0(x) = 1,\ L_1(x) = x - 1$ and

$$L_{n+1}(x) = (x - (2n+1))L_n(x) - n^2 L_{n-1}(x) \quad (n \geqslant 1)$$

(take $\alpha = \gamma = 1$ and $\beta = 2$ in the last theorem). They are the *normalized Laguerre polynomials*

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k}(n)_k(-1)^k x^{n-k}.$$

# References

[1] M. AIGNER, *A Characterization of the Bell Numbers,* Discrete Mathematics 205 (1999), p.207-210.

[2] P. DELSARTE, *Nombres de Bell et polynômes de Charlier,* Comptes Rendus Acad. Sc. 287 série A (1978), p.271-273.

[3] R. EHRENBORG, *The Hankel Determinant of Exponential Polynomials,* American Math. Monthly 107 (2000), p.557-560.

[4] C. RADOUX, *Calcul effectif de certains déterminants de Hankel,* Bull. Soc. Math. Belg. série B31 (1979), p.49-55.

[5] C. RADOUX, *Addition Formulas and Hankel Determinants for some classical Sequences of Integers,* Journal of Comput. and Applied Math. 115 (2000), p.471-477.

[6] A. ROBERT, *Systèmes de Polynômes,* Queen's Papers in Pure and Applied Mathematics, no 35 (1973).

Received: 27. 05. 2002
Revised: 16. 07. 2002